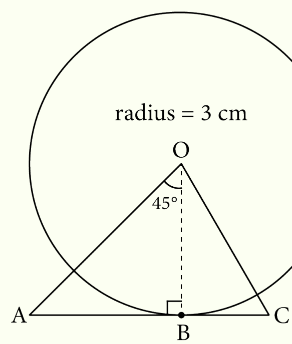
Grade 7/8 Math Circles
February 6/7/8/9, 2023
Circle Geometry - Problem Set

Exercise Solutions

Exercise 1

Given that $\overline{AC} = 5$ cm, find the value of $\overline{BC}$ using your knowledge of circles and triangles.



Solution

First, notice that $\overline{BO}$ is the radius of the circle, which means that the length of $\overline{BO}$ is 3 cm. Then, notice that the base of the triangle is tangent to the circle. This means that it is at a 90° angle to the radius. Since the sum of the interior angles of a triangle is 180°, we can subtract 90° and 45° from 180° to get that $\angle BAO = 45°$. This makes $\triangle ABO$ an isosceles triangle with a 90° angle and two 45° angles. Because $\triangle ABO$ is an isosceles triangle, we know that $\overline{AB} = \overline{BO} = 3$ cm. Since we also know that $\overline{AC} = 5$ cm and that $\overline{AC} = \overline{AB} + \overline{BC}$, we get that:

$$\overline{AC} = \overline{AB} + \overline{BC}$$
$$\overline{BC} = \overline{AC} - \overline{AB}$$
$$\overline{BC} = 5 \text{ cm} - 3 \text{ cm}$$
$$\overline{BC} = 2 \text{ cm}$$

Exercise 2

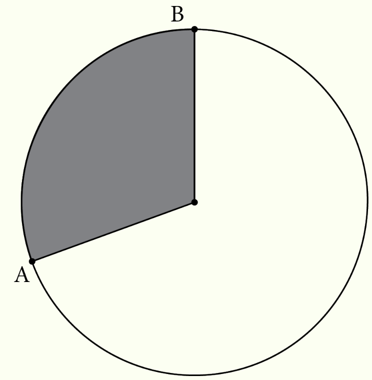
1. Given two chords of equal length, which theorem might you apply?
   (a) Theorem 1 (Radius Bisects Chord)
   (b) Theorem 2 (Intersecting Chords Theorem)
   (c) Theorem 3 (Equal Chords and Equidistant from Center Theorem)
   (d) None of the above
2. Given that a chord bisects another, which theorem might you apply?
   (a) Theorem 1 (Radius Bisects Chord)
   (b) Theorem 2 (Intersecting Chords Theorem)
   (c) Theorem 3 (Equal Chords and Equidistant from Center Theorem)
   (d) None of the above
3. Given two chords that intersect at their end point, which theorem might you apply?
   (a) Theorem 1 (Radius Bisects Chord)
   (b) Theorem 2 (Intersecting Chords Theorem)
   (c) Theorem 3 (Equal Chords and Equidistant from Center Theorem)
   (d) None of the above

Solution

1. (c) 2. (a) or (b) *(Both are actually applicable in this case!)* 3. (d)

Exercise 3

Given that the area of the shaded region is a third of the total area, what would the central angle of $\widehat{AB}$ (the arc between A and B) be? If the radius of the circle is 9 m, what would the length of $\widehat{AB}$ be?



Note that we will use $\widehat{AB}$ to denote the length of the arc between A and B going forward.

Solution

Since the area of the shaded region is a third of the total area, the central angle of $\widehat{AB}$ must also be a third of the circle's central angle. So $\theta = \frac{1}{3} \times 360° = 120°$.

Using the equation for arc length, we get:

$$\begin{aligned}
\widehat{AB} &= \frac{\theta}{360°} \times 2\pi r \\
&= \frac{120°}{360°} \times 2\pi(9 \text{ m}) \\
&= \frac{1}{3} \times 18\pi \text{ m} \\
&= 6\pi \text{ m} \\
&\approx 18.85 \text{ m}
\end{aligned}$$

Notice that you could have skipped the calculation of the angle, since $\frac{\theta}{360°}$ is a calculation for the proportion of the full circle and the proportion of the shaded region was already given.

Exercise 4

1. Given an angle and a radius, which theorem might you apply?
   (a) Theorem 4 (Chord Central Angle Theorem)
   (b) Theorem 5 (Chord Arcs Theorem)

(c) Theorem 6 (Parallel Chords Intercepted Arcs Theorem)

(d) None of the above

2. Given two parallel chords, which theorem might you apply?

   (a) Theorem 4 (Chord Central Angle Theorem)

   (b) Theorem 5 (Chord Arcs Theorem)

   (c) Theorem 6 (Parallel Chords Intercepted Arcs Theorem)

   (d) None of the above

3. Given two chords of the same length, which theorem might you apply?

   (a) Theorem 4 (Chord Central Angle Theorem)

   (b) Theorem 5 (Chord Arcs Theorem)

   (c) Theorem 6 (Parallel Chords Intercepted Arcs Theorem)

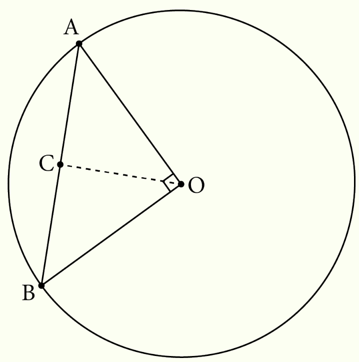
   (d) None of the above

Solution

1. (d) 2. (c) 3. (a) or (b) *(Both can actually be applied in this case!)*

Exercise 5

In the following diagram, $\angle AOB = 90°$. Given that $\overset{\frown}{AB} = 2.5$ cm, find the length of $\overline{CO}$, given that $\overline{CO}$ is perpendicular to $\overline{AB}$.



Solution

Looking at the given information, we know the central angle and arc length of $\overset{\frown}{AB}$. Looking at both the arc length formula and the chord length formula, we can see that we only have enough known information to solve for radius using the arc length formula. So substituting $\theta = 90°$ and $\overset{\frown}{AB} = 2.5$ cm, we have:

$$\begin{aligned}
2.5 \text{ cm} &= \frac{90°}{360°} \times 2\pi r \\
r &= 2.5 \text{ cm} \times \frac{360°}{90°} \times \frac{1}{2\pi} \\
r &= 2.5 \text{ cm} \times 4 \times \frac{1}{2\pi} \\
r &= \frac{5}{\pi} \text{ cm} \\
r &\approx 1.59 \text{ cm}
\end{aligned}$$

Since we have the radius and know that $\triangle ABO$ is an isosceles right angle triangle, we can calculate the length of the chord $\overline{AB}$ by the Pythagorean Theorem.

$$\begin{aligned}
\overline{AB} &= \sqrt{\overline{AO}^2 + \overline{BO}^2} \\
&= \sqrt{2r^2} \\
&= \sqrt{2\left(\frac{5}{\pi}\right)^2} \text{ cm} \\
&= \frac{5\sqrt{2}}{\pi} \text{ cm} \\
&\approx 2.25 \text{ cm}
\end{aligned}$$

Then using the chord length, we can finally solve for $\overline{CO}$ using the chord length formula, which is the perpendicular distance between the chord and the origin.

$$
\begin{aligned}
r^2 &= d^2 + \left(\frac{1}{2}c\right)^2 \\
d^2 &= r^2 - \left(\frac{1}{2}c\right)^2 \\
\overline{CO} &= \sqrt{r^2 - \left(\frac{1}{2}\overline{AB}\right)^2} \\
\overline{CO} &= \sqrt{\left(\frac{5}{\pi}\right)^2 - \left(\frac{1}{2}\frac{5\sqrt{2}}{\pi}\right)^2} \text{ cm} \\
\overline{CO} &= \sqrt{\frac{5^2}{\pi^2} - \frac{5^2}{2\pi^2}} \text{ cm} \\
\overline{CO} &= \sqrt{\frac{5^2}{2\pi^2}} \text{ cm} \\
\overline{CO} &= \frac{5}{\sqrt{2}\pi} \text{ cm} \\
\overline{CO} &\approx 1.13 \text{ cm}
\end{aligned}
$$

Therefore, the length of the line between C and O is $\frac{5}{\sqrt{2}\pi}$ cm.

Problem Set Solutions

1. Explain in your own words what the relationship between the circumference and radius is.

   Solution: Solutions may vary. Since the equation for the circumference of a circle is $C = 2\pi r$, the solution should relate the circumference and radius using 2π.

2. Calculate the radius of a circle if the angle between two lines from the origin to the perimeter is 36° and the arc length created by those lines is 15. Answer in exact form.

   Solution:
   We are given $\theta = 36°$ and $L = 15$. We know that the equation for the arc length is

$L = \frac{\theta}{360°} \times 2\pi r$. So we isolate for r, substitute, and solve.

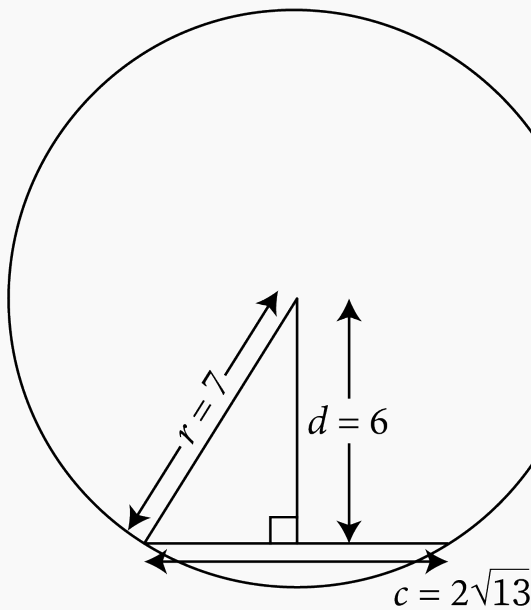
$$\begin{aligned} L &= \frac{\theta}{360°} \times 2\pi r \\ r &= L \times \frac{360°}{\theta} \times \frac{1}{2\pi} \\ r &= (15)\left(\frac{360°}{36°}\right)\left(\frac{1}{2\pi}\right) \\ r &= \frac{(15)(10)}{2\pi} \\ r &= \frac{75}{\pi} \\ r &\approx 23.87 \end{aligned}$$

3. Calculate the length of a chord given that the radius of its circle is 7 and the perpendicular distance from the origin to the chord is 6. Answer in exact form. Draw a diagram and label the chord, radius, and perpendicular distance from the origin.

Solution:
We are given $r = 7$ and $d = 6$. We know that the equation for the length of a chord is $c = 2\sqrt{r^2 - d^2}$. So we substitute and solve for c.

$$\begin{aligned} c &= 2\sqrt{r^2 - d^2} \\ &= 2\sqrt{7^2 - 6^2} \\ &= 2\sqrt{49 - 36} \\ &= 2\sqrt{13} \end{aligned}$$

The diagram should look like the one below.



4. Your classmate says that they drew a circle with a radius of 5 m and a chord with a length of 11 m. Is what your classmate saying possible? Why or why not?

Solution:
Thinking of a circle, the largest chord a circle can have is one through its origin. In other words, the largest chord a circle can have is actually its diameter. Since we know that the diameter of a circle is twice its radius, we know that the largest chord your classmate can draw in a circle with a radius of 5 m is 10 m. Since 11 is larger than 10, your classmate's claim is impossible.

Another (more difficult) way to verify this is by analyzing the equation of the length of a chord. We know that the equation for the length of a chord is $c = 2\sqrt{r^2 - d^2}$. Rearranging and substituting these numbers into the equation, we get:

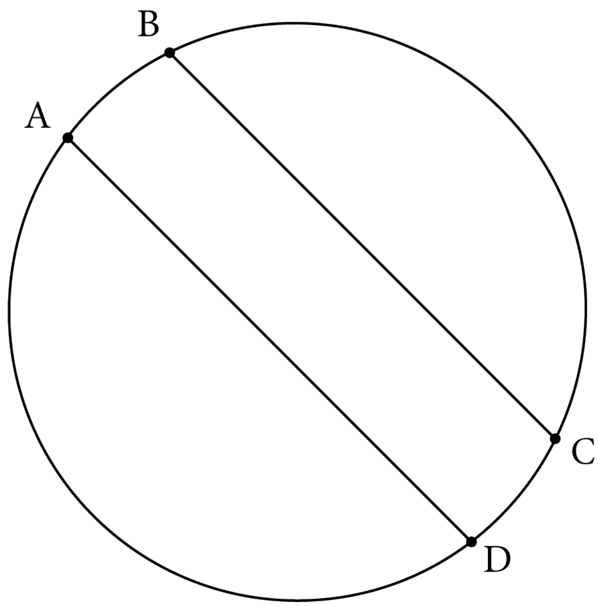
$$c = 2\sqrt{r^2 - d^2}$$
$$r^2 = \left(\frac{1}{2}c\right)^2 + d^2$$
$$5^2 = \left(\frac{11}{2}\right)^2 + d^2$$
$$5^2 = 5.5^2 + d^2$$

We know that if you square a number, it must be positive (or at least 0), which means that d^2 must be positive (or at least 0). Since $5^2 < 5.5^2$, we can reason that $5^2 < 5.5^2 \leq 5.5^2 + d^2$. This means that 5^2 will never equal $5.5^2 + d^2$ and that your classmate's claim is impossible.

5. What is the maximum that an arc length can be in terms of the radius r? Explain your answer briefly.

Solution: We know that the equation for the arc length is $L = \frac{\theta}{360°} \times 2\pi r$. Since the angle is the only variable we are changing, it makes sense to make θ as large as possible to make $\frac{\theta}{360°}$ as large as possible. The largest central angle we can take is 360°, which makes the arc length equal to the circumference or that $L = 2\pi r$.

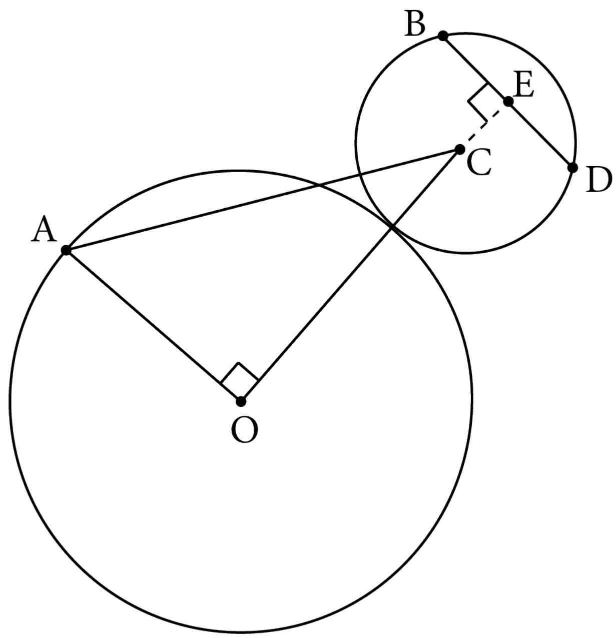
6. Suppose that $\widehat{BC} = 15\pi$, $\widehat{AB} = 3\pi$, and $\overline{AD} \parallel \overline{BC}$. What is $\widehat{AD}$?



Solution: Since $\overline{AD}$ and $\overline{BC}$ are parallel, we can apply **Theorem 6** to conclude that $\overset{\frown}{AB} = \overset{\frown}{CD} = 3\pi$. Since $\overset{\frown}{AD} = \overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD}$, we have the following:

$$\begin{aligned}
\overset{\frown}{AD} &= \overset{\frown}{AB} + \overset{\frown}{BC} + \overset{\frown}{CD} \\
&= 2\overset{\frown}{AB} + \overset{\frown}{BC} \\
&= 2(3\pi) + (15\pi) \\
&= 6\pi + 15\pi \\
&= 21\pi
\end{aligned}$$

7. Suppose $\overline{OC}$ joins the origins of the two circles. Given that $\overline{OC} = 8$, $\overline{AC} = 10$, and $\overline{CE} = 1$, find the chord length of $\overline{BD}$. Answer in exact form.



Solution:
To find $\overline{BD}$, we need to apply the equation for the length of a chord. We are already given the perpendicular distance, $\overline{CE} = 1$, so we need to find the radius of the smaller circle. Notice that since $\overline{OC}$ connects the origins of the two circles, we can see that $\overline{OC}$ = (radius of big circle) + (radius of small circle). Then notice that $\overline{AO}$ is the radius of the big circle and $\overline{BC}$ is the radius of the small circle. This means that $\overline{OC} = \overline{AO} + \overline{BC}$ $(\star)$. We do not have $\overline{AO}$, but we can solve for this by using the Pythagorean Theorem because $\triangle ACO$ is a right angled triangle. So:

$$\overline{AC}^2 = \overline{AO}^2 + \overline{OC}^2$$
$$\overline{AO} = \sqrt{\overline{AC}^2 - \overline{OC}^2}$$
$$\overline{AO} = \sqrt{10^2 - 8^2}$$
$$\overline{AO} = \sqrt{100 - 64}$$
$$\overline{AO} = \sqrt{36}$$
$$\overline{AO} = 6$$

Rearranging for $\overline{BC}$ and substituting $\overline{OC} = 8$ and $\overline{AO}$ into ($\star$) gives us:

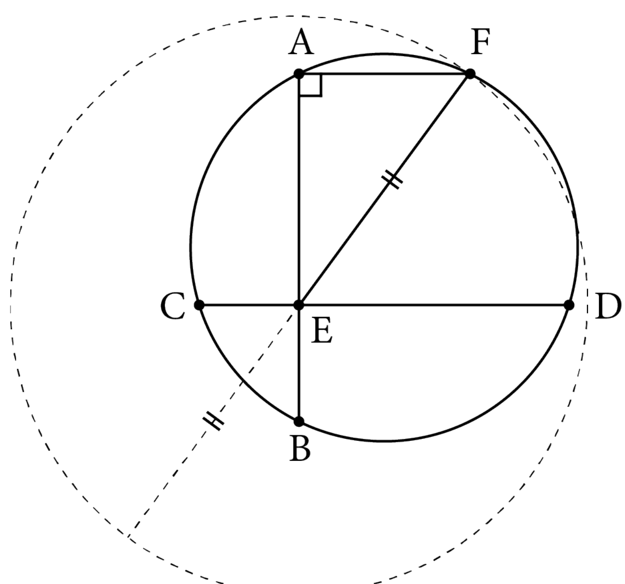
$$\begin{aligned}
\overline{OC} &= (\text{radius of big circle}) + (\text{radius of small circle}) \\
\overline{OC} &= \overline{AO} + \overline{BC} \\
\overline{BC} &= \overline{OC} - \overline{AO} \\
\overline{BC} &= 8 - 6 \\
\overline{BC} &= 2
\end{aligned}$$

Finally, we apply the equation for the length of a chord with $d = \overline{CE} = 1$ and $r = \overline{BC} = 2$, we have:

$$\begin{aligned}
c &= 2\sqrt{r^2 - d^2} \\
\overline{BD} &= 2\sqrt{\overline{BC}^2 - \overline{CE}^2} \\
\overline{BD} &= 2\sqrt{2^2 - 1^2} \\
\overline{BD} &= 2\sqrt{3}
\end{aligned}$$

Therefore, the chord length of $\overline{BD} = 2\sqrt{3}$.

8. Suppose the large dotted circle has a radius of 5 and E is its centre. Suppose $\overline{CE} = 1$, $\overline{DE} = 4.5$, and $\overline{BE} = \frac{9}{8}$. What is the length of $\overline{AF}$?



Solution:
First, notice that since E is the centre of the large dotted circle, $\overline{EF}$ is the radius of the large circle. So this means $\overline{EF} = 5$. Notice that since $\triangle AEF$ is right angled, we can use the Pythagorean Theorem to solve for $\overline{AF}$. But this requires $\overline{AE}$, which we can solve for using **Theorem 2**, which says $\overline{AE} \times \overline{BE} = \overline{CE} \times \overline{DE}$. Rearranging for $\overline{AE}$ and Substituting in $\overline{CE} = 1$, $\overline{DE} = \frac{9}{2}$, and $\overline{BE} = \frac{9}{8}$ gives us:

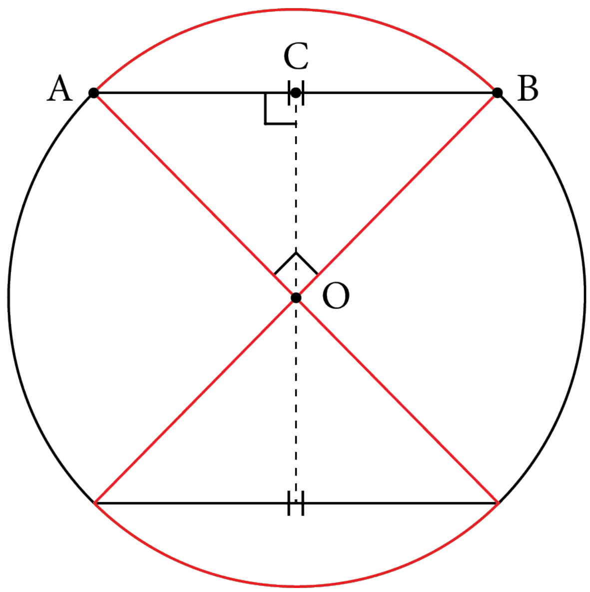
$$\begin{aligned}
\overline{AE} \times \overline{BE} &= \overline{CE} \times \overline{DE} \\
\overline{AE} &= \frac{\overline{CE} \times \overline{DE}}{\overline{BE}} \\
\overline{AE} &= \frac{(1)\left(\frac{9}{2}\right)}{\left(\frac{9}{8}\right)} \\
\overline{AE} &= \left(\frac{9}{2}\right)\left(\frac{8}{9}\right) \\
\overline{AE} &= 4
\end{aligned}$$

Now that we have $\overline{AE} = 4$, we can solve for $\overline{AF}$.

$$\begin{aligned}
\overline{EF}^2 &= \overline{AE}^2 + \overline{AF}^2 \\
\overline{AF} &= \sqrt{\overline{EF}^2 - \overline{AE}^2} \\
\overline{AF} &= \sqrt{5^2 - 4^2} \\
\overline{AF} &= \sqrt{25 - 16} \\
\overline{AF} &= \sqrt{9} \\
\overline{AF} &= 3
\end{aligned}$$

Therefore $\overline{AF} = 3$.

9. Suppose that Alan wanted to paint the region marked in red on his wall. He measured $\overline{AB}$ and $\angle AOC$ and found them to be 2 m long and 45° respectively. What is the total perimeter that he has to paint?



Solution:
Notice that $\angle AOC = 45°$ and $\angle ACO = 90°$. Since the sum of interior angles of a triangle must be $180°$, we can deduce that $\angle OAC = 45°$. This means that $\triangle ACO$ is isosceles, and $\overline{AC} = \overline{CO}$. By **Theorem 1**, $\overline{CO}$ bisects $\overline{AB}$, which means that $\overline{AC} = \frac{1}{2}\overline{AB} = 1$ m. Since $\triangle ACO$ is right angled, we can use the Pythagorean Theorem.

$$\overline{AO}^2 = \overline{AC}^2 + \overline{CO}^2$$
$$\overline{AO} = \sqrt{\overline{AC}^2 + \overline{CO}^2}$$
$$\overline{AO} = \sqrt{2\overline{AC}^2}$$
$$\overline{AO} = \sqrt{2(1 \text{ m})^2}$$
$$\overline{AO} = \sqrt{2} \text{ m}$$

Notice that $\overline{AO}$ is the radius of the circle. So we can apply the formula for the arc of a circle with $r = \overline{AO} = \sqrt{2}$ m and $\theta = \angle AOB = 90°$.

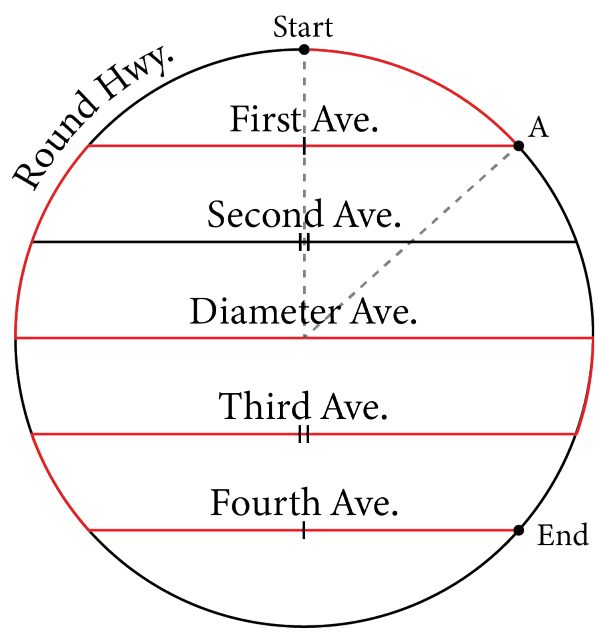
$$\begin{aligned}\overset{\frown}{AB} &= \frac{\theta}{360°} \times 2\pi r \\ &= \frac{90°}{360°} \times 2\pi\overline{AO} \\ &= \frac{1}{4} \times 2\pi\sqrt{2} \text{ m} \\ &= \frac{\pi\sqrt{2}}{2} \text{ m}\end{aligned}$$

Challenge Question

10. A mail man needs to deliver mail throughout Colossal Circle City. It just so happens that Second Ave. has no mail, so he only has to travel the route highlighted in red. Suppose that the following are true:

- First Ave. is $80\sqrt{5} \approx 178.89$ km long
- Second Ave. is $160\sqrt{2} \approx 226.27$ km long
- Diameter Ave. is 240 km long and is the diameter of Round Highway
- The distance travelled on Round St. between First and Second Ave. is 15π km long
- The distance travelled on Round St. between Second and Diameter Ave. is 13π km long
- The proportion of the circumference that the arc from start to point A is $\frac{2}{15}$
- All avenues are parallel to each other

What is the total distance travelled by the mail man (rounded to 2 decimal places)?



Solution:
In order to find the total distance travelled, we need to add all the red arc lengths plus all the lengths of the avenues travelled. The distances we are missing are:

- Third Ave.
- Fourth Ave.

- Arc length from the start to point A
- Arc length between Diameter Ave. and Third Ave.
- Arc length between Third Ave. and Fourth Ave.

First, calculate the arc length from the start to point A. We are given that the arc from start to point A is $\frac{2}{15}$ of the total circumference. We can use this instead of $\frac{\theta}{360°}$ because this fraction gives us the proportion. Since Diameter Ave. is actually the diameter of the city, we can deduce that the radius of the city is half of that, or 120 km. Applying this to the equation for the length of an arc, we have:

$$\begin{aligned}\text{arc from Start to A} &= (\text{proportion of circumference}) \times 2\pi r \\ &= \frac{2}{15} \times 2\pi 120 \text{ km} \\ &= 32\pi \text{ km}\end{aligned}$$

It is shown on the diagram that the length of Third Ave. is equal to the length of Second Ave., and the length of Fourth Ave. is equal to the length of First Ave. Then, notice that Colossal Circle City is symmetrical about Diameter Ave. This means that the top half of the city can vertically flipped along Diameter Ave. to make the bottom half of the city. Since all the avenues are parallel to each other, we can conclude that the arc length between First Ave. and Second Ave. is equal on both sides by **Theorem 6**. Combining this with the idea that Colossal Circle City is symmetrical about Diameter Ave, we can reason that the arc length between First Ave. and Second Ave. is also equal to the arc length between Third Ave. and Fourth Ave. This reasoning can also be applied to the arc length between Second Ave. and Diameter Ave. and the arc length between Diameter Ave. and Third Ave. Let a denote the total distance travelled on the arc lengths. So, the sum of the arc lengths is:

$$\begin{aligned}
a =&(\text{arc from Start to A}) + (\text{arc between First and Second Ave.}) +\\
&(\text{arc between Second and Diameter Ave.}) + (\text{arc between Diameter and Third Ave.}) +\\
&(\text{arc between Third and Fourth Ave.})\\
=&(\text{arc from Start to A}) + 2 \times (\text{arc between First and Second Ave.}) +\\
&2 \times (\text{arc between Second and Diameter Ave.})\\
=&32\pi \text{ km} + 2(15\pi \text{ km}) + 2(13\pi \text{ km})\\
=&88\pi \text{ km}\\
\approx&276.46 \text{ km}
\end{aligned}$$

Then let d denote the total distance travelled on the avenues. The sum of the distance travelled on the avenues is:

$$\begin{aligned}
d &= (\text{First Ave.}) + (\text{Diameter Ave.}) + (\text{Third Ave.}) + (\text{Fourth Ave.})\\
&= 2 \times (\text{First Ave.}) + (\text{Diameter Ave.}) + (\text{Second Ave.})\\
&= 2(80\sqrt{5} \text{ km}) + 240 \text{ km} + 80\sqrt{2} \text{ km}\\
&\approx 2(178.89) + 240 + 226.27 \text{ km}\\
&\approx 824.05 \text{ km}
\end{aligned}$$

Finally, we add a and d to find the total distance travelled by the mail man.

$$\begin{aligned}
a + d &\approx 276.46 \text{ km} + 824.05 \text{ km}\\
&\approx 1100.51 \text{ km}
\end{aligned}$$

So the mail man travelled about 1100.51 km.